# TESTS FOR CONVERGENCE OR DIVERGENCE OF SERIES

If an infinite series converges, then an infinite number of terms adds up to something finite. Otherwise, the series diverges. There are many tests to determine the convergence and divergence of series. They are stated below:

1) Given a series $\sum_{n=1}^{\infty} a_n$, if $\lim_{n\to\infty} a_n \neq 0$, then the series **diverges**.

2) Given a series $\sum_{n=1}^{\infty} a_n$, if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, then the series **converges**. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, then the series **diverges**. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, then the series **may converge or diverge**. This test is known as the ***Ratio Test***.

3) Another test that is related to the ratio test is the ***Root Test***. The test states that if $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$, then the series **converges**. The test states that if $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$, then the series **diverges**. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then the series **may converge or diverge**.

4) The ***Comparison Test*** is the most obvious of the tests. It states that if $0 \le a_n \le b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ **converges**, and if $0 \le a_n \le b_n$ and $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ **diverges**.

5) The ***Integral Test*** states that if $\int_1^{\infty} a_n dn$ is finite, then $\sum_{n=1}^{\infty} a_n$ **converges**. Otherwise, the series $\sum_{n=1}^{\infty} a_n$ **diverges**.

6) The ***Alternating Series Test*** states that if $\lim_{n\to\infty} b_n = 0$, $b_n \ge b_{n+1}$ for all n, and the series $\sum_{n=1}^{\infty} a_n$ alternates in sign where $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n b_n$ then the series $\sum_{n=1}^{\infty} a_n$ **converges**.

7) Another test, which comes from the comparison test, is the ***Absolute Convergence Test***. It states that if $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ **converges**.

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8) Another test is to apply the monotone convergence theorem for sequences. If it can be shown that the sequence of partial sums $\left\{\sum_{n=1}^{m} a_n\right\}_{m=1}^{\infty}$ are monotone and bounded, then the series $\sum_{n=1}^{\infty} a_n$ **converges**.

9) If $\lim_{n\to\infty} \frac{b_n}{a_n} = k > 0$, then both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ **converge** or they both **diverge**. This is known as the ***second comparison test*** or ***limit comparison test***.(For non-alternating series)

***Definition***: If $\sum_{n=1}^{\infty} a_n$ **converges**, but $\sum_{n=1}^{\infty} |a_n|$ **diverges**, then we say that the series $\sum_{n=1}^{\infty} a_n$ is ***conditionally convergent***.

## Special Series:

Theorem: If $p > 1$, then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ **converges**, and if $p \leq 1$ then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ **diverges**. This series is known as the ***p-series***.

Theorem: The series $\sum_{n=0}^{\infty} x^n$ **converges** if $|x| < 1$ and diverges if $|x| \geq 1$. This series is known as a ***geometric series***. If the series converges, then $\sum_{n=0}^{\infty} x^n$ is equal to $\frac{1}{1-x}$.