GRAPHING LINES

In high school algebra, as well as in other math courses, one has to graph lines. Many things in various science courses are linear, and graphing lines is done in many science courses. In mathematics, a line can be recognized because the exponents on the variables "**x**" and "**y**" equal to **1**. The below examples illustrate this:

Example 1: $2x + 3y = 1$ ←This is a line because the exponents on the "**x**" and the "**y**" both equal **1**.

Example 2: $2x^3 + 3y^2 = 1$ ←This is not a line because the exponent on the "**x**" = **3** and the exponent on the "**y**" = **2**.

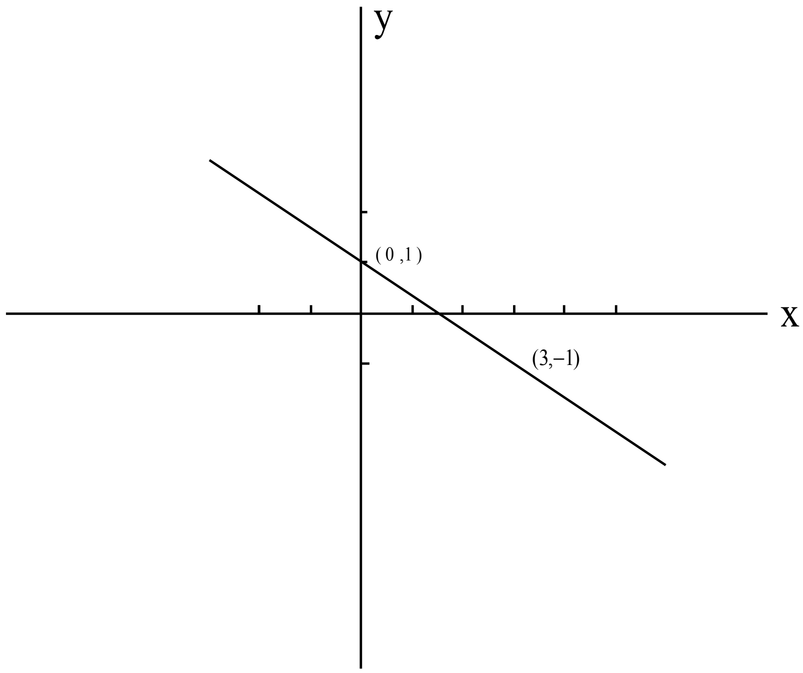
In order to graph a line, two important pieces of information are needed: the slope and the y-intercept. The **<u>slope</u>** is the measure of how "steep" a line is. The formula for the slope is $\mathbf{m} = \frac{\mathbf{y}_2 - \mathbf{y}_1}{\mathbf{x}_2 - \mathbf{x}_1}$ where $(\mathbf{x}_1, \mathbf{y}_1)$ and $(\mathbf{x}_2, \mathbf{y}_2)$ are points on the line. The slope is referred to as "the **rise** over the **run**". The **<u>y-intercept</u>** is where the line will cross the y-axis. To find these values, one must put the line in the slope-intercept form. The **<u>slope-intercept form</u>** of a line is $y = mx + b$ where **m** = the slope of the line, and **b** = the y-intercept of the line. The below examples illustrate slopes, y-intercepts, and the slope-intercept form of a line:

Example: Find the slope and y-intercept of the line $2\mathbf{x} + 3\mathbf{y} = 3$. First, we solve for "y" as shown at right:

$$\begin{array}{r} 2x + 3y = 3 \\ -2x \qquad\quad -2x \\ \hline 3y = -2x + 3 \end{array} \Rightarrow \tfrac{1}{3}(3y = -2x + 3) \Rightarrow y = \tfrac{-2x}{3} + 1$$

The above equation shows that $\mathbf{m} = -2/3$ and $\mathbf{b} = 1 =$ y-intercept. We can use this to graph the line. Since $m = -2/3$, the "rise" equals **–2** and the "run" equals **3**. We can also say that the "rise" equals **2** and the "run" equals **–3** because $\mathbf{m} = -2/3 = 2/-3$. Given a point **A** on the line, the "rise" is how many units we go up from **A**. If the "rise" is negative, then we go down from **A**. The "run" is how many units we go right from point **A**. If the "run" is negative, then we go to the left from point **A**.

The slope and y-intercept are used to graph the above line as follows: Since $\mathbf{b} = 1 =$ y-intercept, then **(0,1)** is a point on the line. Since $\mathbf{m} = -2/3$, the "rise" equals **–2**, and the "run" equals **3**; we go down 2 units from **(0,1)** bringing us to **(0, –1)** and we go to the right three units from **(0, –1)** bringing us to **(3, –1)**. Hence **(0,1)** and **(3, –1)** are points on the line. To get more points on the line, repeat the procedure in the previous two sentences. We graph the points **(0,1)** and **(3, –1)** and connect them with a ruler to graph the line. It is highly recommended that one graphs three points and see if they fall on a line to check for errors. The above line is graphed below:



Sometimes, we are given two points on the line, and we need to find the slope **m** of the line. The below example illustrates this:

<u>Example 2</u>: Find the slope of the line that connects the points **(0,4)** and **(2, –2)**. We use the formula for the slope $\mathbf{m} = \frac{\mathbf{y}_2 - \mathbf{y}_1}{\mathbf{x}_2 - \mathbf{x}_1}$ where $(\mathbf{x}_1, \mathbf{y}_1)$ and $(\mathbf{x}_2, \mathbf{y}_2)$ are points on the line. Letting $\mathbf{x}_1 = 0; \mathbf{y}_1 = 4; \mathbf{x}_2 = 2; \mathbf{y}_2 = -2$ we have $\mathbf{m} = \frac{-2-4}{2-0} = \frac{-6}{2} = -3$.

In this example and in any other example, it does not matter whether we have $\mathbf{x}_1 = 0; \mathbf{y}_1 = 4; \mathbf{x}_2 = 2; \mathbf{y}_2 = -2$ or $\mathbf{x}_1 = 2; \mathbf{y}_1 = -2; \mathbf{x}_2 = 0; \mathbf{y}_2 = 4$. The result will come out the same. Note that if $\mathbf{x}_1 = 2; \mathbf{y}_1 = -2; \mathbf{x}_2 = 0; \mathbf{y}_2 = 4$ then $\mathbf{m} = \frac{4-(-2)}{0-2} = \frac{6}{-2} = -3$.

It is possible that in the formula $\mathbf{m} = \frac{\mathbf{y}_2 - \mathbf{y}_1}{\mathbf{x}_2 - \mathbf{x}_1}$, we could have a zero in the denominator or $\mathbf{x}_2 - \mathbf{x}_1 = 0$. In that case, the line is a **vertical line** with undefined slope.

There is another form of a line which is used when one is given the slope of the line and a point of that line. It is called the <u>**point-slope form**</u> of a line which is illustrated below: $(\mathbf{y} - \mathbf{y}_1) = \mathbf{m}(\mathbf{x} - \mathbf{x}_1)$ where **m** = the slope and $(\mathbf{x}_1, \mathbf{y}_1)$ is a point on the line. The below example illustrates the use of the point-slope form:

Example 3: Find the equation of the line that has slope 2 and goes through $(1,3)$. Substituting $m = 2$, $\mathbf{x}_1 = 1$, **and** $\mathbf{y}_1 = 3$ we have $(\mathbf{y} - 3) = 2(\mathbf{x} - 1)$.

Another use of the point-slope form of a line is when one is given two points on a line and has to find the equation of that line. The below example illustrates this:

Example 4(Extension of Example 2): Find the equation of the line that connects the points **(0,4)** and **(2, –2)**. From example 2, we found that $\mathbf{m} = \frac{-2-4}{2-0} = \frac{-6}{2} = -3$. Substituting in the equation $(\mathbf{y} - \mathbf{y}_1) = \mathbf{m}(\mathbf{x} - \mathbf{x}_1)$, we have $(\mathbf{y} + 2) = -3(\mathbf{x} - 2)$. Also, $(\mathbf{y} - 4) = -3(\mathbf{x} - 0) \Rightarrow \mathbf{y} - 4 = -3\mathbf{x}$ is the equation as well. It does not matter whether we pick **(2, –2)** as $(\mathbf{x}_1, \mathbf{y}_1)$ or whether we pick **(0,4)** as $(\mathbf{x}_1, \mathbf{y}_1)$. We will get the same line.

Another form of a line is the **two-intercept form** of a line which is illustrated below: $\frac{\mathbf{x}}{\mathbf{a}} + \frac{\mathbf{y}}{\mathbf{b}} = 1$ where **a** = the x-intercept and **b** = the y-intercept. This form is not valid for vertical lines because vertical lines are of the form x = **c** and vertical lines have no y-intercepts. The form is also not valid for horizontal lines because horizontal lines are of the form y = **d** and horizontal lines have no x-intercepts. This form is useful but is not taught in some high school and college math courses.

If two lines intersect each other at right angles(90° angles), then they are **perpendicular**. Perpendicular lines have slopes that are negative reciprocals of each other. To put the previous sentence another way, if line **A** has slope **m** and line **B** has slope **n**, then the product **m·n** = **–1** or $\mathbf{n} = {}^{-1}\!/_{\mathbf{m}}$. The following example illustrates the above concept:

Example 5: Find the line that is perpendicular to $2\mathbf{x} + 3\mathbf{y} = 3$ and goes through the point **(2,4)**. From example 1, the slope $\mathbf{m} = {}^{-2}\!/_{3}$. The reciprocal of **m** is ${}^{-3}\!/_{2}$, and the negative reciprocal of **m** is ${}^{3}\!/_{2}$. Substituting into the point-slope form of a line, we have $(\mathbf{y} - 4) = {}^{3}\!/_{2}(\mathbf{x} - 2)$.

There is only one exception to the above rule concerning perpendicularity: Any horizontal line and any vertical line are perpendicular, but a vertical line has undefined slope.

Given any two lines. There are three possibilities concerning their intersection: They intersect at one point; they are parallel(The lines do not intersect.); or they are the same line. <u>Parallel lines always have the same slope</u>. If the lines are the same, they obviously have the same slope. The below example illustrates finding a line parallel to another line(See the next page.):

Example 6: Find the line that is parallel to $2\mathbf{x}+3\mathbf{y}=3$ and goes through the point **(2,4)**. From example 3, the slope $\mathbf{m}={}^{-2}\!/_{3}$. The line that we are looking for has the same slope. Substituting into the point-slope form of a line, we have $(\mathbf{y}-4)={}^{-2}\!/_{3}(\mathbf{x}-2)$.

The following example illustrates what happens when two lines are the same line. The two lines have an infinite number of solutions that will satisfy both of the lines:

Example 7: Find a solution that will satisfy both $2\mathbf{x}+3\mathbf{y}=3$ and $4\mathbf{x}+6\mathbf{y}=6$. Solving for y in the first equation $2\mathbf{x}+3\mathbf{y}=3$ as was done in example 3, we have $\mathbf{y}={}^{-2}\!/_{3}\,\mathbf{x}+1$. Substituting $\mathbf{y}={}^{-2}\!/_{3}\,\mathbf{x}+1$ into $4\mathbf{x}+6\mathbf{y}=6$, we have the following below:
$4\mathbf{x}+6({}^{-2}\!/_{3}\,\mathbf{x}+1)=6\Rightarrow 4\mathbf{x}-4\mathbf{x}+6=6\Rightarrow 6=6\Rightarrow$ **The lines are the same**. Any solution that will work in the first line will work in the second line. One such solution for the first line is **(0,1)**. Note that it will work in $4\mathbf{x}+6\mathbf{y}=6$ because $4(0)+6(1)=6$. If one is observant, one will notice that $4\mathbf{x}+6\mathbf{y}=6$ **is** $2(2\mathbf{x}+3\mathbf{y}=3)$. This always happens when two lines are the same. To get another solution of $2\mathbf{x}+3\mathbf{y}=3$, pick a value of "**x**" and solve for "**y**". It will work for the second line as well.