LIMITS:

One of the most important topics in calculus is the topic of **limits**. Many people who take calculus often confuse the concept of a limit thinking that all limits are direct "plug-ins" in f(x) or they don't fully understand what a limit tells them about a function f(x). The symbols "$\lim_{x \to a} f(x)$" mean as **x** gets closer to "**a**" what is f(x) getting closer to, **NOT** what f(x) is at **a**.

A non-mathematical analogy of a limit is a lake in a desert. As one approaches the location of the lake, what is one approaching? Either the lake is real or it is a mirage. The lake or the mirage is the limit while the location is "**a**".

There are many different situations that one encounters when evaluating limits. Each situation will be illustrated below with an example so that the student will obtain a high degree of confidence evaluating limits:

Example 1(*Evaluating a limit when f(x) is a polynomial*): Let $f(x) = 2x^2 + 4$ and let $x \to 1$; in other words, evaluate $\lim_{x \to 1} (2x^2 + 4)$. One way to evaluate this limit is to pick values close to 1 and see what $2x^2 + 4$ gets closer to. Pick the values .9, .95, .99, .995 The results are below:

$f(x) := 2x^2 + 4$
$f(.9) = 5.62$
$f(.95) = 5.805$
$f(.99) = 5.96$
$f(.995) = 5.98$

> As one can clearly see, the values of f(x) are approaching the value 6.

Pick the values 1.1, 1.05, 1.01, 1.005 The results are below:

$f(x) := 2x^2 + 4$
$f(1.1) = 6.42$
$f(1.05) = 6.205$
$f(1.01) = 6.04$
$f(1.005) = 6.02$

> As one can clearly see, the values of f(x) are approaching the value 6 again.

Hence, $\lim_{x \to 1} (2\mathbf{x}^2 + 4) = 6$. In this problem, one could have plugged in 6 and gotten the same result; however, not all limits can be evaluated by plug-ins. One of the reasons that the above limit was a direct plug-in was that f(x) is a polynomial. ***In general, if f(x) is a polynomial, then the limit is a direct plug-in.*** The next example will show that a limit is not a direct plug-in. Evaluating limits by picking values close to "**a**", while not wrong, can be extremely cumbersome.

Example 2(*Evaluating a limit when f(x) is a rational function*): Let $\mathbf{f}(\mathbf{x}) = \frac{\mathbf{x}^2 - 4}{\mathbf{x} - 2}$ and let $\mathbf{x} \to 2$; in other words, evaluate $\lim_{x \to 2} \frac{\mathbf{x}^2 - 4}{\mathbf{x} - 2}$. Directly plugging in **x** = 2 will yield a zero in the denominator and one may erroneously conclude that $\mathbf{f}(\mathbf{x}) = \frac{\mathbf{x}^2 - 4}{\mathbf{x} - 2}$ has no limit. As $\mathbf{x} \to 2$, keep in mind that **x** ≠ 2 because we are approaching 2. We're not at 2**!** As long as **x** ≠ 2, one can factor the numerator so that $\mathbf{f}(\mathbf{x}) = \frac{(\mathbf{x} - 2)(\mathbf{x} + 2)}{(\mathbf{x} - 2)}$ and cancel the (**x** – 2)'s in the numerator and the denominator yielding $\mathbf{f}(\mathbf{x}) = \mathbf{x} + 2$. Hence,

$$\lim_{x \to 2} \frac{\mathbf{x}^2 - 4}{\mathbf{x} - 2} = \lim_{x \to 2} (\mathbf{x} + 2) = 4.$$

Note that if the function f(x) cannot be simplified allowing cancellation or simplification as was done above, then the limit may not exist. Note that $\lim_{x \to 2} \frac{\mathbf{x}^2 + \mathbf{x} + 1}{\mathbf{x} - 2}$ does not have a limit because $(\mathbf{x} + 2)$ goes to zero as $\mathbf{x} \to 2$; $\mathbf{x}^2 + \mathbf{x} + 1$ is not factorable; and no cancellation or simplification results. *(Note that* $\mathbf{x}^2 + \mathbf{x} + 1$ *has complex roots. College freshman and sophomore calculus courses do not cover the topic of complex numbers.)*

Limits need to be evaluated when f(x) is a piecewise function. A <u>piecewise function</u> is a function that is "pasted together" or "grafted" as shown in the below example. There is an analog to a piecewise function in biology. One gets a branch of a tree and pastes it on another tree yielding a grafted tree.

Example 3(*Evaluating a limit when f(x) is a piecewise function*): Let $\mathbf{f}(\mathbf{x}) = \begin{cases} \mathbf{x} + 2 & \textbf{if} \quad \mathbf{x} \geq 2 \\ 2\mathbf{x} & \textbf{if} \quad \mathbf{x} < 2 \end{cases}$. Note that f(x) has two "branches". One branch is for **x** < 2, and the other branch is for **x** ≥ 2.

When one evaluates the limit of f(x) as $\mathbf{x} \to 2$, one has to be careful. When $\mathbf{x} \to 2$, there are two possibilities: One can approach **x =** 2 from the left of **x =** 2, or one can approach **x =** 2 from the right of **x =** 2. In order to consider the two possibilities, we must evaluate the "one-sided limits" in order to evaluate $\lim_{\mathbf{x}\to 2} \mathbf{f}(\mathbf{x})$.

The special notation for the "one-sided limits" are as follows: $\lim_{\mathbf{x}\to \mathbf{a}^+} \mathbf{f}(\mathbf{x})$ = the limit of f(x) from the right side of "**a**", and $\lim_{\mathbf{x}\to \mathbf{a}^-} \mathbf{f}(\mathbf{x})$ = the limit of f(x) from the left side of "**a**". **Both limits must exist and be equal in order for** $\lim_{\mathbf{x}\to \mathbf{a}} \mathbf{f}(\mathbf{x})$ **to exist**.

Applying the above concept to example 3, we have $\lim_{\mathbf{x}\to 2^+} \mathbf{f}(\mathbf{x})$ = the limit of f(x) from the right side of **2**, and $\lim_{\mathbf{x}\to 2^-} \mathbf{f}(\mathbf{x})$ = the limit of f(x) from the left side of **2**. Let's evaluate the one-sided limit $\lim_{\mathbf{x}\to 2^+} \mathbf{f}(\mathbf{x})$. As $\mathbf{x} \to 2^+$, we are considering values of **x** that are to the right of **2** but
not equal to **2**; hence, **x > 2**. We must determine what branch of the piecewise function f(x) we are in. Since **x > 2**, **x** ≥ 2 is also true; hence, f(x) = **x** + **2** and we are in the first branch of the piecewise function f(x). Since f(x) = **x** + **2**, then $\lim_{\mathbf{x}\to 2^+} \mathbf{f}(\mathbf{x}) = \lim_{\mathbf{x}\to 2^+} (\mathbf{x}+2) = 4$. Since (**x** + **2**) is a polynomial, then $\lim_{\mathbf{x}\to 2^+} (\mathbf{x}+2)$ is a direct plug-in as explained in example 1.

Let's evaluate $\lim_{\mathbf{x}\to 2^-} \mathbf{f}(\mathbf{x})$ = the limit of f(x) from the left side of **2**. As $\mathbf{x} \to 2^-$, we are considering values of **x** that are to the left of **2** but not equal to **2**; hence, **x < 2**. We must determine what branch of the piecewise function f(x) we are in. Since **x < 2**, f(x) = **2**x and we are in the second branch of the piecewise function f(x). Since f(x) = **2**x, then $\lim_{\mathbf{x}\to 2^-} \mathbf{f}(\mathbf{x}) = \lim_{\mathbf{x}\to 2^-} 2\mathbf{x} = 4$. Since **2**x is a polynomial, then $\lim_{\mathbf{x}\to 2^-} 2\mathbf{x}$ is a direct plug-in as explained in example 1.

Since $\lim_{\mathbf{x}\to 2^-} \mathbf{f}(\mathbf{x}) = \lim_{\mathbf{x}\to 2^-} 2\mathbf{x} = 4$ and $\lim_{\mathbf{x}\to 2^+} \mathbf{f}(\mathbf{x}) = \lim_{\mathbf{x}\to 2^+} (\mathbf{x}+2) = 4$, then $\lim_{\mathbf{x}\to 2} \mathbf{f}(\mathbf{x}) = 4$.

Notice in example 3, **x** = **2** acted as a "cut point" of f(x); in other words, on one side of $\mathbf{x} = 2$, f(x) = **2**x and on the other side of $\mathbf{x} = 2$, f(x) = **x** + **2**. Let's consider the same function $\mathbf{f}(\mathbf{x}) = \begin{cases} \mathbf{x}+2 & \textbf{if} \quad \mathbf{x} \geq 2 \\ 2\mathbf{x} & \textbf{if} \quad \mathbf{x} < 2 \end{cases}$ but $\lim_{\mathbf{x}\to 3} \mathbf{f}(\mathbf{x})$ will be evaluated instead.

<u>Example 4</u>(*Evaluating a limit when f(x) is a piecewise function but not at the "cut point".*):

Once again, let $\mathbf{f(x)} = \begin{cases} \mathbf{x}+2 & \textbf{if} \quad \mathbf{x} \geq 2 \\ 2\mathbf{x} & \textbf{if} \quad \mathbf{x} < 2 \end{cases}$ but evaluate $\lim_{\mathbf{x}\to 3} \mathbf{f(x)}$.

Applying the concept of one-sided limits to example 4, we have $\lim_{\mathbf{x}\to 3^+} \mathbf{f(x)}$ = the limit of f(x) from the right side of **3**, and $\lim_{\mathbf{x}\to 3^-} \mathbf{f(x)}$ = the limit of f(x) from the left side of **3**. Let's evaluate the one-sided limit $\lim_{\mathbf{x}\to 3^+} \mathbf{f(x)}$. As $\mathbf{x} \to 3^+$, we are considering values of **x** that are to the right of **3** but not equal to **3**; hence, **x > 3**. We must determine what branch of the piecewise function f(x) we are in. Since **x > 3**, **x ≥ 2** is also true; hence, f(x) = **x** + **2** and we are in the first branch of the piecewise function f(x). Since f(x) = **x** + **2**, then $\lim_{\mathbf{x}\to 3^+} \mathbf{f(x)} = \lim_{\mathbf{x}\to 3^+} (\mathbf{x}+2) = 5$. Since (**x** + **2**) is a polynomial, then $\lim_{\mathbf{x}\to 3^+} (\mathbf{x}+2)$ is a direct plug-in as explained in example 1.

<u>Example 4</u>(Continued) Let's evaluate $\lim_{\mathbf{x}\to 3^-} \mathbf{f(x)}$ = the limit of f(x) from the left side of **3**. As $\mathbf{x} \to 3^-$, we are considering values of **x** that are to the left of **3** but not equal to **3**; hence, **x < 3**. We must determine what branch of the piecewise function f(x) we are in. As was done in example 1, we only consider values of **x** that are very close to and to the left of **3** such as **2.9**, **2.99**, **2.999**. That is what $\mathbf{x} \to 3^-$ means. Hence, we are still in the first branch of f(x) namely, **(x + 2)**. Since f(x) = **x** + **2**, then $\lim_{\mathbf{x}\to 3^-} \mathbf{f(x)} = \lim_{\mathbf{x}\to 3^-} (\mathbf{x}+2) = 5$. Because (**x** + **2**) is a polynomial, $\lim_{\mathbf{x}\to 3^-} (\mathbf{x}+2)$ is a direct plug-in as explained in example 1. The final result is $\lim_{\mathbf{x}\to 3} \mathbf{f(x)} = 5$.

The previous two examples showed a piecewise function where the one-sided limits existed and were equal. The reader should not get the impression that limits exist for all piecewise functions f(x) for every value of **x**. The next example will show this:

<u>Example 5</u>(*Evaluating a limit when f(x) is a piecewise function*): Let $\mathbf{f(x)} = \begin{cases} \mathbf{x}+1 & \textbf{if} \quad \mathbf{x} \geq 3 \\ 2\mathbf{x} & \textbf{if} \quad \mathbf{x} < 3 \end{cases}$ and evaluate $\lim_{\mathbf{x}\to 3} \mathbf{f(x)}$. Once again, we evaluate the one-sided limits to determine if $\lim_{\mathbf{x}\to 3} \mathbf{f(x)}$ exists. Applying the concept of one-sided limits to example 5, we have $\lim_{\mathbf{x}\to 3^+} \mathbf{f(x)}$ = the limit of f(x) from the right side of **3**, and $\lim_{\mathbf{x}\to 3^-} \mathbf{f(x)}$ = the limit of f(x) from the left side of **3**. Let's evaluate the one-sided limit $\lim_{\mathbf{x}\to 3^+} \mathbf{f(x)}$. As $\mathbf{x} \to 3^+$, we are considering values of **x** that are to the right of **3** but not equal to **3**; hence, **x > 3**. We

must determine what branch of the piecewise function f(x) we are in. Since **x > 3**, **x ≥ 3** is also true; hence, f(x) = **x** + **1** and we are in the first branch of the piecewise function f(x). Since f(x) = **x** + **1**, then $\lim_{x\to 3^+} \mathbf{f}(\mathbf{x}) = \lim_{x\to 3^+}(\mathbf{x}+1) = 4$. Since (**x** + **1**) is a polynomial, then $\lim_{x\to 3^+}(\mathbf{x}+1)$ is a direct plug-in as explained in example 1.

Let's evaluate $\lim_{x\to 3^-} \mathbf{f}(\mathbf{x})$ = the limit of f(x) from the left side of **3**. As $\mathbf{x} \to 3^-$, we are considering values of **x** that are to the left of **3** but not equal to **3**; hence, **x < 3**. We must determine what branch of the piecewise function f(x) we are in. Since **x < 3**, f(x) = **2**x and we are in the second branch of the piecewise function f(x). Since f(x) = **2**x, then $\lim_{x\to 3^-} \mathbf{f}(\mathbf{x}) = \lim_{x\to 3^-} 2\mathbf{x} = 6$. Since **2**x is a polynomial, then $\lim_{x\to 2^-} 2\mathbf{x}$ is a direct plug-in as explained in example 1.

Since $\lim_{x\to 3^-} \mathbf{f}(\mathbf{x}) = \lim_{x\to 3^-} 2\mathbf{x} = 6$ and $\lim_{x\to 3^+} \mathbf{f}(\mathbf{x}) = \lim_{x\to 3^+}(\mathbf{x}+1) = 4$, then $\lim_{x\to 2} \mathbf{f}(\mathbf{x})$ **<u>does not exist</u>**.

<u>WHAT RADICALS & FRACTIONS DO TO LIMITS</u>:

Note that fractions and radicals impose restrictions on how one can evaluate limits. We have already seen that having a zero in the denominator can lead to having no limit. (*Remember the discussion about* $\lim_{x\to 2} \frac{\mathbf{x}^2+\mathbf{x}+1}{\mathbf{x}-2}$ *on page **2**!*). Radicals impose the restriction that we cannot take the square root of a negative number. In fact, if the index number of a radical is even, then we cannot have a negative radicand. For odd index numbers, there is no restriction. Note that we can take the cube root of any negative number. For those who forgot, the parts of a radical expression are below:

$$\text{index number} \to \sqrt[n]{\mathbf{x}} \leftarrow \textbf{radicand(What's inside the radical.)}$$
$$\uparrow \textbf{radical(funny looking "division sign")}$$

The next example shows what happens when we have to deal with radicals and limits:

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<u>Example 6</u>(*Evaluating a limit when f(x) has a radical in it*): Evaluate $\lim_{x\to 2} \frac{4}{\sqrt{\mathbf{x}-2}+4}$. It is very tempting to just plug in **2** and say the limit is $\frac{4}{\sqrt{2-2}+4} = \frac{4}{4} = 1$. What is being

asked here is the two sided limit as $x \to 2$. Does the two-sided limit exist? The answer is **NO!** In order for the two sided limit as $x \to 2$ to exist, we must approach **x = 2** from both sides of **2**. Note that if $x \to 2^-$, **x < 2**. If **x < 2**, then the radicand **x – 2 < 0**. The index number on the radical is an understood **2** which is an even number. Since the index number is even, then we cannot have a negative radicand. Hence,

$\lim_{x \to 2^-} \frac{4}{\sqrt{x-2}+4}$ does not exist.

<u>Example 7</u>(*Changing the index number in example 6*): Evaluate $\lim_{x \to 2} \frac{4}{\sqrt[3]{x-2}+4}$. Does the two-sided limit exist? In order for the two sided limit as $x \to 2$ to exist, we must approach $x = 2$ from both sides of **2**. Note that if $x \to 2^-$, **x < 2**. If **x < 2**, then the radicand **x – 2 < 0**. The index number on the radical = **3** which is an odd number. Since the index number is odd, there is no restriction on the radicand. Hence,

$\lim_{x \to 2^-} \frac{4}{\sqrt{x-2}+4} = \frac{4}{4} = 1$. Also, $\lim_{x \to 2^+} \frac{4}{\sqrt{x-2}+4} = \frac{4}{4} = 1$. Hence, $\lim_{x \to 2} \frac{4}{\sqrt{x-2}+4} = \frac{4}{4} = 1$.

The lesson of examples 6 and 7 is to check the domain of the function before evaluating the limit! Changing the index numbers of the radicals in a limit problem can make a <u>big difference</u>!