Limit Problem for Discussion

Prove that $\lim_{x \to 4} x^2 = 16$

Proof for the above problem:

For all $\varepsilon > 0$, we must find a $\delta > 0$ where $0 < |x - a| < \delta$ implies $|f(x) - L| < \varepsilon$. In this problem; $L = 16$, $f(x) = x^2$, $a = 4$.

In particular, we must find a $\delta > 0$ where $0 < |x - 4| < \delta$ implies $|x^2 - 16| < \varepsilon$. Note that $|f(x) - L| = |x^2 - 16| = |x - 4| \cdot |x + 4|$.

Also, if we let $|x - 4| < 1$, we have $-1 < x - 4 < 1$. (Always work on the "$|x - 4|$" part and choose 1. Any number other than 1 can be chosen.) The previous sentence implies that $7 < x + 4 < 9$ (Just add 8 to $-1 < x - 4 < 1$.) Hence, $|x + 4| < 9$.

Note that $|x^2 - 16| = |x - 4| \cdot |x + 4| < 9|x - 4|$. Setting $9|x - 4| < \varepsilon$ we have $|x - 4| < \varepsilon/9$.

Letting $\delta \le \min(1, \varepsilon/9)$, we have $0 < |x - 4| < \delta$ implies $|x^2 - 16| < \varepsilon$.

Actually, I proved that $f(x) = x^2$ is continuous at $x = 4$ above, but continuity implies the assertion above. For polynomial functions $f(x)$, it is easier to prove continuity using the delta(δ)-epsilon(ε) definition below:

For all $\varepsilon > 0$, we must find a $\delta > 0$ where $|x - a| < \delta$ implies $|f(x) - L| < \varepsilon$. and use that to say the limit exists.